Matrix coefficient realization theory of noncommutative rational functions

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Associated areas

- Theory of skew fields: universal construction
- Theoretical computer science: weighted finite automata
- Free real algebraic geometry: linear matrix inequalities
- Systems/control theory: linear systems evolving on a free group
- Noncommutative symmetric functions: quasi-determinants

Nc rational expressions

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Evaluations on matrices:

- $\mathcal{M} = \bigcup_{m\in\mathbb{N}} M_m(\mathbb{F})^g$.
- $\operatorname{dom} r$ is the subset of $\mathcal{M}$ where $r \in \mathcal{R}_{\mathbb{F}}(Z)$ is defined; $\operatorname{dom}_m r = \operatorname{dom} r \cap M_m(\mathbb{F})^g$.
- r is **degenerate** if $\operatorname{dom} r = \emptyset$ and **nondegenerate** otherwise.
- For nondegenerate r_1 and r_2: $r_1 \sim r_2$ iff $r_1(a) = r_2(a)$ for all $a \in \operatorname{dom} r_1 \cap \operatorname{dom} r_2$.

Nc rational functions

Nc rational functions are equivalence classes of nondegenerate expressions, $\mathbb{F}\not<Z\not>$. This is a skew field (with obvious operations).
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We can also describe $\mathbb{F}\langle\!\langle Z \rangle\!\rangle$ using

- rational expressions over an ∞-dim skew field (Amitsur, Bergman),
- full matrices over $\mathbb{F}<Z>$ (Cohn),
- Malcev-Neumann construction on a free group (Lewin),
- skew field associated to a free Lie algebra (Lichtman).

Universal property

$\mathbb{F}(\!<Z>\!)$ is **the universal skew field of fractions** of $\mathbb{F}<Z>$, i.e., for every skew field $D \supseteq \mathbb{F}$ and epimorphism $\mathbb{F}<Z> \to D$ we have a commutative diagram

$$\begin{array}{ccccc} \mathbb{F}<Z> & \hookrightarrow & K & \hookrightarrow & \mathbb{F}(\!<Z>\!) \\ & \searrow & \downarrow{\scriptstyle \phi} & & \\ & & D & & \end{array}$$

where K is a local ring and $\phi : K \to D$ satisfies $\phi(x) \neq 0 \Rightarrow x^{-1} \in K$.

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Informally: if a nc rational expression vanishes on all tuples of matrices over $\mathbb{F}$, then it vanishes on all tuples of elements in D, where D is an arbitrary skew field containing $\mathbb{F}$.
Such expressions are called **rational identities**.

Functions analytic at the origin

It can be hard to distinguish nc rational functions; e.g.

$$\left(z_1 - (1+z_2)^{-1} z_1 (1+z_2)\right) \left((1+z_2)^{-1} z_1 (1+z_2) - z_2^{-1} z_1 z_2^{-1}\right)^{-1} - z_2$$

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Let $r \in \mathcal{R}_{\mathbb{F}}(Z)$. If $(0,\dots,0) \in \operatorname{dom} r$, then we can expand r into a noncommutative power series $S \in \mathbb{F}\ll Z \gg$. Such a series is **rational**, i.e., it belongs to the rational closure of $\mathbb{F}<Z>$ in $\mathbb{F}\ll Z\gg$.

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Theorem (Schützenberger, '61)

Every rational series S has a **linear representation**, *i.e., there exists $n \in \mathbb{N}$ and $\mathbf{c} \in \mathbb{F}^{1\times n}$, $\mathbf{b} \in \mathbb{F}^{n\times 1}$ and $A_j \in \mathbb{F}^{n\times n}$ for $1 \le j \le g$, such that*

$$S=\mathbf{c}\left(I_n-\sum_{j=1}^{g}A_jz_j\right)^{-1}\mathbf{b}.$$

General case

Previous result can be applied to rational expressions defined at some point in $\mathbb{F}^g$. What about other rational expressions, e.g. $(z_1 z_2 - z_2 z_1)^{-1}$?

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Let $m \in \mathbb{N}$. The algebra of **generalized polynomials over** $M_m(\mathbb{F})$ is defined as

$$M_m(\mathbb{F})<Z> := M_m(\mathbb{F}) *_{\mathbb{F}} \mathbb{F}<Z>\,.$$

Its (Z)-completion is called the algebra of **generalized series over** $M_m(\mathbb{F})$ and denoted $M_m(\mathbb{F})\ll Z\gg$.

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Assume r is defined at $p \in M_m(\mathbb{F})^g$. Then we can expand r into a generalized series S about the point p. Again, this series belongs to the rational closure of $M_m(\mathbb{F})\!<\!Z\!>$ in $M_m(\mathbb{F})\!\ll\!Z\!\gg$.

Realizations

If $S \in M_m(\mathbb{F})\ll Z \gg$ is a rational series, then there exists $n \in \mathbb{N}$ and

$$\mathbf{c} \in M_m(\mathbb{F})^{1\times n}, \quad \mathbf{b} \in M_m(\mathbb{F})^{n\times 1}, \quad A_j \in \sum M_m(\mathbb{F})^{n\times n} z_j M_m(\mathbb{F})^{n\times n}$$

for $1 \le j \le g$, such that

$$S = \mathbf{c}\left(I_n - \sum_{j=1}^{g} A_j\right)^{-1} \mathbf{b}.$$

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If $r \in \mathcal{R}_{\mathbb{F}}(Z)$ is defined at $p \in M_m(\mathbb{F})^g$ and S is its expansion about p, then $(\mathbf{b}, A, \mathbf{c})$ is called a **realization of** r **about** p of dimension n.

Interpretation

Let $(\mathbf{b}, A, \mathbf{c})$ be a realization of r about p and let $s \in \mathbb{N}$.
With some abuse of the notation we can write

$$\mathbf{c}\left(I_{nms} - \sum_{j=1}^{g} A_j(q_j - p_j)\right)^{-1}\mathbf{b} = r(q) \in M_{ms}(\mathbb{F})$$

for $q \in M_{ms}(\mathbb{F})^g$, where the entries of $\mathbf{c}$, $\mathbf{b}$ and A_j are considered as elements in $M_{ms}(\mathbb{F})$ using the embedding

$$M_m(\mathbb{F}) \hookrightarrow M_{ms}(\mathbb{F}), \qquad a \mapsto \begin{pmatrix} a & & \\ & \ddots & \\ & & a \end{pmatrix}.$$

Two examples

Realization of $((1 - z_1 - z_2(1 - z_1)^{-1} z_2)^{-1}$ about $(0, 0)$:

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \left(I_2 - \begin{pmatrix} z_1 & 0 \\ 0 & z_1 \end{pmatrix} - \begin{pmatrix} 0 & z_2 \\ z_2 & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Realization of $(z_1 z_2 - z_2 z_1)^{-1}$ about $(p_1, p_2) \in M_2(\mathbb{F})^2$ assuming that $p_1 p_2 - p_2 p_1$ is invertible with inverse q:

$$\mathbf{c} \Big(I_3 - A_1(z_1 - p_1) - A_2(z_2 - p_2) \Big)^{-1} \mathbf{b}, \quad \text{where}$$

$$\mathbf{c} = \begin{pmatrix} q & 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -z_1 p_2 q + p_2 z_1 q & z_1 & 0 \\ 0 & 0 & 0 \\ -z_1 q & 0 & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} z_2 p_1 q - p_1 z_2 q & 0 & -z_2 \\ -z_2 q & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Special types of representations

Let $(\mathbf{b}, A, \mathbf{c})$ be a representation over $M_m(\mathbb{F})$ of dimension n of series S. Its **obstruction modules** are

$$\mathcal{U}_L = \{\mathbf{u} \in M_m(\mathbb{F})^{1\times n} : \mathbf{u}A_{i_1} \dots A_{i_\ell}\mathbf{b} = 0 \ \forall i_j, \ell\},$$
$$\mathcal{U}_R = \{\mathbf{u} \in M_m(\mathbb{F})^{n\times 1} : \mathbf{c}A_{i_1} \dots A_{i_\ell}\mathbf{u} = 0 \ \forall i_j, \ell\}.$$

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We say that $(\mathbf{b}, A, \mathbf{c})$ is

- **reduced** if $\mathcal{U}_L$ and $\mathcal{U}_R$ are torsion $M_m(\mathbb{F})$-modules;
- **minimal** if its dimension is minimal amongst all representations of S;
- **totally reduced** if if $\mathcal{U}_L$ and $\mathcal{U}_R$ are trivial.

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4. The dimension of a reduced representation is greater than the minimal one for at most 1.
5. A totally reduced representation is unique up to a basis change.
6. For a rational expression and "almost every" point in its domain, we can find its totally reduced realization using the previously mentioned algorithm.

Degree of nc rational function

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Therefore we can define **the degree** of a nc rational function. It is independent of the choice of a representing expression $r \in \mathbb{r}$, point of expansion p, and even the size of the matrices in p.

Domain of a nc rational function

$$r_1 = (1 - z_1 - z_2(1 - z_1)^{-1}z_2)^{-1},$$
$$r_2 = -z_2^{-1}(1 - z_1)(z_2 - (1 - z_1)z_2^{-1}(1 - z_1))^{-1},$$
$$r_3 = \begin{pmatrix} 1 & 0 \end{pmatrix} \left(I_2 - \begin{pmatrix} z_1 & 0 \\ 0 & z_1 \end{pmatrix} - \begin{pmatrix} 0 & z_2 \\ z_2 & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

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It can be shown: $\mathbb{r}_1 = \mathbb{r}_2 = \mathbb{r}_3$, $(1, 1) \in \operatorname{dom} r_2 \setminus \operatorname{dom} r_1$, $(0, 0) \in \operatorname{dom} r_1 \setminus \operatorname{dom} r_2$ and $\operatorname{dom} r_3 \supsetneq \operatorname{dom} r_1 \cup \operatorname{dom} r_2$.

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Theorem

If $(\mathbf{c}, A, \mathbf{b})$ is a totally reduced realization of $\mathbb{r}$ about $p \in M_m(\mathbb{F})^g$, then

$$\operatorname{dom}_{ms} \mathbb{r} = \left\{ q \in M_{ms}^g(\mathbb{F}) \colon \det \left(I_{nms} - \sum_{j=1}^{g} A_j (q_j - p_j) \right) \neq 0 \right\}.$$

The rational identity testing problem

Denote

$$\kappa(r) = \#\text{constant\_terms\_in\_}r + 2 \cdot \#\text{letters\_in\_}r + \#\text{inversions\_in\_}r.$$

Example: for

$$r = 2(z_2 z_1 - 1)^{-1} - (z_1 z_2^{-1} - z_2^{-1} z_1)^{-1} - 1)$$

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Theorem

Let $r \in \mathcal{R}_{\mathbb{F}}(Z)$ and assume $\operatorname{dom}_m r \neq \emptyset$. Then r is a rational identity if and only if r vanishes on $\operatorname{dom}_N r$, where

$$N = m \left\lceil \frac{m}{2} \kappa(r) \right\rceil .$$

... and another version

Well known: if $p \in \mathbb{F}{<}Z{>} \setminus \{0\}$ vanishes on $M_N(\mathbb{F})^g$, then the (polynomial) degree of p is at least $2N$.

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